Goldstein 9.24 (a)

$$Q(p,q) = p + iaq \quad , \quad P(p,q) = \frac{p - iaq}{2ia}$$

$$2iaP = p - iaq$$

$$\Rightarrow \frac{1}{2}\left[Q + 2iaP\right] = p(Q,P)$$

$$\frac{1}{2ia}\left[Q - 2iaP\right] = q(Q,P)$$

Now that we have $Q(p,q)$, $P(p,q)$, $q(P,Q)$, $p(P,Q)$, we wish to check relation

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P} \quad \text{and} \quad \left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P}$$

(Goldstein 9.48)

$$\left(\frac{\partial Q}{\partial q}\right)_{q,p} = ia, \quad \left(\frac{\partial p}{\partial P}\right)_{Q,P} = ia$$

$$\left(\frac{\partial Q}{\partial p}\right)_{q,p} = 1, \quad -\left(\frac{\partial q}{\partial P}\right)_{Q,P} = 1$$

To find the generating function, we seek a representation of two of p, q, P, Q in the other two coordinates, one that works is

$$p = iaq + 2iaP, \quad Q = 2iaq + 2iaP$$

This represents (p, Q) in (P, q), and suggests

$$\boxed{F_2 = \frac{iaq^2}{2} + 2iaPq + iaP^2}$$

Goldstein 9.24 (b)

The classical SHO is given by

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$$

we previously found $p(Q,P) = \frac{1}{2}[Q + 2iaP]$,

$$q(Q,P) = \frac{1}{2ia}[Q - 2iaP],$$

This suggests $K(Q,P) = \frac{1}{2m}\frac{1}{4}[Q+2iaP]^2 + \frac{m\omega^2}{2}\frac{1}{(2ia)^2}[Q-2iaP]^2$

$$\cancel{= \frac{1}{8m}[Q^2 - 4a^2Q^2 + 4iaPQ]}$$

$$\cancel{- \frac{m\omega^2}{8a^2}[Q^2 - 4a^2P^2}$$

$$= \frac{1}{8m}[Q^2 - 4a^2P^2 + 4iaPQ] - \frac{m\omega^2}{8a^2}[Q^2 - 4a^2P^2 - 4iaPQ]$$

$$= \frac{1}{8}\left[\frac{Q^2}{m} - \frac{4a^2P^2}{m} + \frac{4ia}{m}PQ - \frac{m\omega^2}{a^2}Q^2 + \frac{4m\omega^2\cancel{a^2}}{\cancel{a^2}}P^2 + \frac{m\omega^2 4ia}{a^2}PQ\right]$$

$$= \frac{1}{8}\left[\left(\frac{1}{m} - \frac{m\omega^2}{a^2}\right)Q^2 + \left(4m\omega^2 - \frac{4a^2}{m}\right)P^2 + \left(\frac{4ia}{m} + \frac{4im\omega^2}{a}\right)PQ\right]$$

$$\frac{\partial K}{\partial Q} = \frac{1}{8}\left[2\left(\frac{1}{m} - \frac{m\omega^2}{a^2}\right)Q + \left(\frac{4ia}{m} + \frac{4im\omega^2}{a}\right)P\right]$$

$$\frac{\partial K}{\partial P} = \frac{1}{8}\left[2\left(4m\omega^2 - \frac{4a^2}{m}\right)P + \left(\frac{4ia}{m} + \frac{4im\omega^2}{a}\right)Q\right]$$

Introduce $\alpha \equiv \frac{1}{4}\left(\frac{1}{m} - \frac{m\omega^2}{a^2}\right)$,

$$\beta \equiv \left(m\omega^2 - \frac{a^2}{m}\right),$$

$$\gamma \equiv \frac{1}{2}\left(\frac{ia}{m} + \frac{im\omega^2}{a}\right),$$

$$\frac{\partial K}{\partial Q} = \alpha Q + \gamma P, \qquad \frac{\partial K}{\partial P} = \beta P + \gamma Q$$

The equation of motion is then

$$\dot{P} = -\alpha Q - \gamma P, \qquad \dot{Q} = \beta P + \gamma Q$$

$$\beta P = \dot{Q} - \gamma Q, \quad P = \frac{\dot{Q}}{\beta} - \frac{\gamma}{\beta} Q, \quad \dot{P} = \frac{\ddot{Q}}{\beta} - \frac{\gamma}{\beta}\dot{Q},$$

$$\frac{\ddot{Q}}{\beta} - \frac{\gamma}{\beta}\dot{Q} = -\alpha Q - \gamma\left[\frac{\dot{Q}}{\beta} - \frac{\gamma}{\beta} Q\right]$$

$$\ddot{Q} - \cancel{\gamma\dot{Q}} = -\alpha\beta Q - \cancel{\gamma\dot{Q}} + \gamma^2 Q$$

$$\ddot{Q} = (\gamma^2 - \alpha\beta)\, Q$$

$$\boxed{Q = A \exp[\eta t], \quad \eta = \pm\sqrt{\gamma^2 - \alpha\beta}}$$

$\Uparrow$

This is the general solution.

we compute $\sqrt{\gamma^2 - \alpha\beta}$

$$\gamma^2 = \frac{1}{4}\left(-\frac{\cancel{\alpha}a^2}{m^2} - \frac{m^2\omega^4}{a^2} - 2\omega^2\right)$$

$$\alpha\beta = \frac{1}{4}\left(\omega^2 - \frac{\alpha^2}{m^2} - \frac{m^2\omega^4}{a^2} + \omega^2\right)$$

$$\sqrt{\gamma^2 - \alpha\beta} = \sqrt{\frac{1}{4}(-4\omega^2)}$$

$$= \pm i\omega$$

$$\Rightarrow \boxed{\eta = \pm i\omega}$$

We get the correct frequency.

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